The Partial Differential Equations for the Hyperelliptic Θ- and σ-Functions.

By Oskar Bolza.

The partial differential equation satisfied by the hyperelliptic Θ- and σ-functions, which furnish the recursion formulæ for the expansion of these functions into power series, have been first established by Wiltheiss in a series of papers published in Crelle's Journal* and the Mathematische Annalen.† Two steps can be distinguished in his deductions:

First he establishes a system of partial differential equations for a canonical system of integrals of the first and second kind or their periods, the differentiation taking place either with respect to the roots or with respect to the coefficients of the polynomial on whose square root the integrals depend.

From these preliminary differential equations he derives, in a second step, the final equations for the Θ-functions by several different methods, all of them indirect and rather complex.

The principal object of the present paper is to replace this part of Wiltheiss' work by simpler and more direct proofs. Two such proofs are given in §4 and §5: the first proceeds directly from Weierstrass' definition of the most general Θ-function by means of an exponential series without using any further properties of the Θ-functions; the second starts from the expression of the Θ-functions in terms of the ϑ-functions, and makes use of the well-known partial differential equations of the ϑ-function. In both proofs the work is vastly simplified by the

---

* Bd. 99, p. 236 (1884).

† Bd. 29, p. 272 (1886); Bd. 31, p. 134 (1887); Bd. 33, p. 267 (1888).

use of the notations and methods of the *theory of matrices** which has already been so successfully applied to the treatment of Θ-functions by Baker in his book on Abelian functions.

The preceding §§1 to 3 contain a simplified proof for the partial differential equations satisfied by the periods of the integrals of the first and second kind. For independent variables I have chosen the branch-points, as Wiltheiss does in his first paper; it seems indeed, from the simple recursion formula given by Brioschi† for the expansion of the even σ-functions of two variables, that differentiation with respect to the branch-points will, in the end, furnish simpler results than the Aronhold process which Wiltheiss uses in his later publications.

The two concluding §§6 and 7 contain applications of the previous results to Wiltheiss' *Th*-functions, to that class of σ-functions for which—in Klein's notation—$\mu = 0$, and to the expression of $\Theta(0, 0 \ldots . 0)$ in terms of the branch-points.

§1.—*Notations.*

I use throughout the notations of Weierstrass' Lectures on Hyperelliptic Functions of Winter 1881–82, with the same slight modifications adopted in my former papers "On Weierstrass' Systems of Hyperelliptic Integrals of the First and Second Kind,"‡ and "On the First and Second Logarithmic Derivatives of Hyperelliptic σ-functions."§

Accordingly the *hyperelliptic curve* is denoted by

$$y^2 = R(x) = \sum_{\lambda=0}^{2\rho+2} \binom{2\rho+2}{\lambda} A_\lambda x^{2\rho+2-\lambda} = A_0 \prod_{i=0}^{2\rho+1} (x - a_i), \tag{1}$$

$$w_\alpha = \int \frac{g_\alpha(x)\, dx}{y} \qquad \alpha = 1, 2, \ldots . , \rho$$

are a system of ρ linearly independent *integrals of the first kind.*

---

*Cayley, "A Memoir on the Theory of Matrices," Phil. Trans., vol. 148. See also Taber, "On the Theory of Matrices," American Journal, vol. XII; Weyr, "Zur Theorie der bilinearen Formen, Monatshefte für Mathematik u. Physik, Bd. 1, and Baker, "Abel's Theorem and the Allied Theory," Art. 189–191, 405–410.

† Göttinger Nachrichten, 1890, p. 236.

‡ Papers read at the International Mathematical Congress, 1893, p. 1.

§ American Journal of Mathematics, vol. XVII, p. 11.

$F(x, \xi)$ is an integral function of x and ξ of degree $\rho+1$ in either variable, satisfying the three conditions:

$$\left.\begin{aligned} F(\xi, x) &= F(x, \xi), \\ F(\xi, \xi) &= R(\xi) \quad , \\ \left(\frac{\partial F(x, \xi)}{\partial x}\right)_{x=\xi} &= \tfrac{1}{2} R'(\xi) \quad , \end{aligned}\right\} \tag{2}$$

$R'(x)$ denoting the derivative of $R(x)$.

The integral

$$\int \frac{F(x, \xi)\, dx}{(x-\xi)^2 y\eta}$$

—where $\eta^2 = R(\xi)$—is of the second kind; its only poles are the two conjugate points $(\xi, \pm\eta)$, and in the vicinity of these poles an expansion of the following form holds:

$$\int \frac{F(x, \xi)\, dx}{(x-\xi)^2 y\eta} = \frac{\mp 1}{x-\xi} + \mathfrak{P}(x-\xi). \tag{3}$$

If a be a branch-point, the integral

$$\int \frac{F(x, a)\, dx}{(a-a)^2 y} \text{ is again of the second kind,}$$

has but one pole a, and admits in its vicinity the expansion

$$\int \frac{F(x, a)\, dx}{(x-a)^2 y} = -\frac{\sqrt{R'(a)}}{(x-a)^{\frac{1}{2}}} + \mathfrak{P}((x-a)^{\frac{1}{2}}). \tag{4}$$

The polynomials of degree $\rho-1$ $g_\alpha(x)$, together with the function $F(x, \xi)$, determine uniquely ρ polynomials $g_{\rho+\alpha}(x)$ of degree 2ρ at most by means of the fundamental relation

$$\sum_\alpha \frac{g_{\rho+\alpha}(\xi) g_\alpha(x)}{\eta} = \frac{d}{d\xi}\left(\tfrac{1}{2}\frac{\eta}{(x-\xi)}\right) - \frac{F(x, \xi)}{2(x-\xi)^2 \eta}. \tag{5}$$

The ρ integrals

$$w_{\rho+\alpha} = \int \frac{g_{\rho+\alpha}(x)\, dx}{y} \qquad \alpha = 1, 2, \ldots, \rho$$

are of the second kind and constitute, together with the ρ integrals w_α, a *canonical system of associated integrals of the first and second kind.*

In the following developments no special assumption concerning this canonical system is made except that we suppose *the polynomials* $g_\alpha(x)$ *to be independent of the branch-points* a_λ.

The letters $\alpha, \beta, \gamma \ldots.$ are always used for summation indices running from 1 to ρ; $\sum\limits_{\alpha,\beta}$ means $\sum\limits_{\alpha=1}^{\rho}\sum\limits_{\beta=1}^{\rho}$, whereas the symbol $\sum\limits_{(\alpha,\beta)}$ denotes summation over all the $\frac{\rho(\rho-1)}{2}$ two-combinations of the numbers $1, 2, \ldots., \rho$.

§2.—*The Derivatives of the Integrals of the First Kind with Respect to a Branch-point.**

Let a be any one of the $2\rho + 2$ branch-points a_λ; we propose in this § to determine the partial derivative of the integral w_α with respect to a.

Following the example of Klein and Wiltheiss, we unite the ρ integrals w_α into one by introducing a parameter ξ as follows: Let $h_1(\xi), h_2(\xi), \ldots., h_\rho(\xi)$ be ρ polynomials in ξ defined by the equation

$$(x-\xi)^{\rho-1} = \sum_\beta g_\beta(x)\, h_\beta(\xi). \tag{6}$$

The $h_\beta(\xi)$'s will be of degree $\rho - 1$ at most, linearly independent, and independent of a. Hence, if the path of integration be likewise independent of a,

$$\sum_\beta h_\beta(\xi)\frac{\partial w_\beta}{\partial a} = \int \frac{\partial}{\partial a}\,\frac{(x-\xi)^{\rho-1}}{y}\,dx = \tfrac{1}{2}\int \frac{(x-\xi)^{\rho-1}\,dx}{(x-a)\,y}.$$

This is an integral of the second kind; in order to express it in terms of the integrals w_α, $w_{\rho+\alpha}$, notice that its only pole is a and the expansion in the vicinity of a:

$$-\frac{(a-\xi)^{\rho-1}(x-a)^{-\frac{1}{2}}}{\sqrt{R'(a)}} + \ldots.$$

Hence, according to (4), the integral

$$\int\left[\tfrac{1}{2}\,\frac{(x-\xi)^{\rho-1}}{x-a} - \frac{(a-\xi)^{\rho-1}}{R'(a)}\,\frac{F(x,a)}{(x-a)^2}\right]\frac{dx}{y}$$

is of the first kind; the expression in the bracket [ ] is therefore a polynomial

---

* Compare for this § also Schroeder, "Ueber den Zusammenhang hyperelliptischer σ- und ϑ-Functionen," Diss. Göttingen, 1890, §9.

in x of degree $\rho - 1$ at most, and the same is true with respect to ξ. We may therefore write

$$\tfrac{1}{2}\frac{(x-\xi)^{\rho-1}}{x-a} - \frac{(a-\xi)^{\rho-1}}{R'(a)}\frac{F(x,\,a)}{(x-a)^2} = \sum_{\alpha,\,\beta} \varkappa_{\alpha\beta}\, g_\alpha(x)\, h_\beta(\xi), \tag{7}$$

the $\varkappa_{\alpha\beta}$'s being independent of x and ξ.

The integral of the second kind,

$$\int \frac{F(x,\,a)\,dx}{(x-a)^2\,y},$$

is at once expressible in terms of the integrals $w_{\rho+\alpha}$ by means of (5), viz.

$$\int \frac{F(x,\,a)\,dx}{(x-a)^2\,y} = \frac{y}{a-x} - 2\sum_\alpha g_\alpha(a)\, w_{\rho+\alpha}.$$

We thus obtain

$$\frac{\partial}{\partial a}\int \frac{(x-\xi)^{\rho-1}\,dx}{y} = \int \Big[\tfrac{1}{2}\frac{(x-\xi)^{\rho-1}}{x-a} - \frac{(a-\xi)^{\rho-1}}{R'(a)}\frac{F(x,\,a)}{(x-a)^2}\Big]\frac{dx}{y}$$
$$- \frac{(a-\xi)^{\rho-1}}{R'(a)}\Big[\frac{y}{a-x} - 2\sum_\alpha g_\alpha(a)\, w_{\rho+\alpha}\Big]. \tag{8}$$

To obtain from (8) the derivatives $\frac{\partial w_\alpha}{\partial a}$, it only remains to arrange, by means of (6) and (7), both sides according to the functions $h_\beta(\xi)$ and equate corresponding coefficients.

If, in particular, we suppose the path of integration to be closed in the Riemann surface, and denote by

$$2\omega_\alpha \text{ and } 2\eta_\alpha,$$

the corresponding periods of the integrals w_α and $w_{\rho+\alpha}$ respectively, we obtain the

Theorem I:

The simultaneous periods $2\omega_\alpha$, $2\eta_\alpha$ of the integrals w_α, $w_{\rho+\alpha}$ satisfy the differential equations

$$\frac{\partial \omega_\alpha}{\partial a} = \sum_\beta \varkappa_{\beta\alpha}\,\omega_\beta - \sum_\beta \frac{2 g_\alpha(a)\, g_\beta(a)}{R'(a)}\,\eta_\beta, \tag{A}$$

the quantities $\varkappa_{\alpha\beta}$ being defined by (6) and (7).

§3.—*The Derivatives of the Integrals of the Second Kind with respect to a Branch-point.*

To obtain the derivatives $\frac{\partial w_{\rho+\alpha}}{\partial a}$ we start from equation (5), in which we interchange x and ξ. Differentiating with respect to a and integrating with respect to x along a path independent of a, we obtain

$$\sum_{\alpha} g_{\alpha}(\xi)\frac{\partial w_{\rho+\alpha}}{\partial a}=\tfrac{1}{2}\frac{\partial}{\partial a}\;\frac{y}{\xi-x}+\int\left[-\tfrac{1}{4}\,\frac{F(x,\,\xi)}{(x-a)(x-\xi)^{2}}-\tfrac{1}{2}\,\frac{\frac{\partial F(x,\,\xi)}{\partial a}}{(x-\xi)^{2}}\right]\frac{dx}{y}. \tag{9}$$

The integral on the right-hand side is of the second kind; its poles are $x=a$ and the two conjugate points $(\xi,\,\pm\eta)$. In the vicinity of a the expansion begins with the term

$$\tfrac{1}{2}\,\frac{F(a,\,\xi)}{(\xi-a)^{2}\sqrt{R'(a)}}\,(x-a)^{-\frac{1}{2}},$$

which is at the same time the first term in the expansion of

$$-\tfrac{1}{2}\,\frac{F(a,\,\xi)}{(\xi-a)^{2}\,R'(a)}\int\frac{F(x,\,a)\,dx}{(x-a)^{2}\,y},$$

according to (4).

In the vicinity of $(\xi,\,\pm\eta)$ the first term of the expansion is

$$\tfrac{1}{4}\,\frac{\mp\eta}{\xi-a}\cdot\frac{1}{x-\xi},$$

since
$$F(x,\,\xi)=R(\xi)+\tfrac{1}{2}R'(\xi)(x-\xi)+\ldots\,,$$
and therefore

$$\frac{\partial F(x,\,\xi)}{\partial a}=\frac{\partial R(\xi)}{\partial a}+\ldots=-\,\frac{R(\xi)}{\xi-a}+\ldots$$

But this is, according to (3), at the same time the first term in the expansion of

$$\frac{1}{4(\xi-a)}\int\frac{F(x,\,\xi)\,dx}{(x-\xi)^{2}\,y}.$$

Hence, if we put for shortness,

$$\Lambda(x,\,\xi)=\tfrac{1}{4}\left(\frac{1}{x-a}+\frac{1}{\xi-a}\right)\frac{F(x,\,\xi)}{(x-\xi)^{2}}+\tfrac{1}{2}\,\frac{\frac{\partial F(x,\,\xi)}{\partial a}}{(x-\xi)^{2}}$$
$$+-\tfrac{1}{2}\,\frac{F(x,\,a)\,F(\xi,\,a)}{R'(a)(x-a)^{2}(\xi-a)^{2}}, \tag{10}$$

the integral
$$\int \frac{\Lambda(x, \xi)\, dx}{y}$$

is of the first kind and consequently $\Lambda(x, \xi)$ a polynomial in x of degree $\rho - 1$ at most, and since
$$\Lambda(x, \xi) = \Lambda(\xi, x), \tag{11}$$

the same is true with respect to the variable ξ. We may therefore write
$$\Lambda(x, \xi) = \sum_{\alpha, \beta} \lambda_{\alpha\beta}\, g_\alpha(x)\, g_\beta(\xi), \tag{12}$$

where $\lambda_{\alpha\beta}$ is independent of x and ξ, and moreover
$$\lambda_{\alpha\beta} = \lambda_{\beta\alpha}. \tag{13}$$

Equation (9) may now be written
$$\sum_\alpha g_\alpha(\xi) \frac{\partial w_{\rho+\alpha}}{\partial a} = \frac{\partial}{\partial a}\left(\tfrac{1}{2}\frac{y}{\xi - x}\right) - \int \frac{\Lambda(x, \xi)\, dx}{y}$$
$$+ \tfrac{1}{4} \int \frac{F(x, \xi)\, dx}{(\xi - a)(x - \xi)^2 y} - \tfrac{1}{2} \int \frac{F(x, a)\, F(\xi, a)\, dx}{R'(a)(x - a)^2 (\xi - a)^2 y},$$

and if we express the two last integrals in terms of the integrals $w_{\rho+\alpha}$ by means of (5) and let the path of integration be the same closed path as in §2, we obtain
$$\sum_\alpha g_\alpha(\xi) \frac{\partial \eta_\alpha}{\partial a} = -\sum_{\alpha, \beta} \lambda_{\alpha\beta}\, g_\beta(\xi)\, w_\alpha - \sum_\beta \eta_\beta \left[\frac{g_\beta(\xi)}{2(\xi - a)} - \frac{F(\xi, a)\, g_\beta(a)}{R'(a)(\xi - a)^2}\right].$$

From (2) it follows that the expression
$$\chi_\beta(\xi) = \frac{g_\beta(\xi)}{2(\xi - a)} - \frac{F(\xi, a)\, g_\beta(a)}{R'(a)(\xi - a)^2}$$

remains finite in $\xi = a$, and is therefore an integral function of ξ of degree $\rho - 1$. In order to arrange it according to the functions $g_\alpha(\xi)$, multiply by $h_\beta(t)$ and sum from $\beta = 1$ to $\beta = \rho$:
$$\sum_\beta \chi_\beta(\xi)\, h_\beta(t) = \frac{(\xi - t)^{\rho-1}}{2(\xi - a)} - \frac{(a - t)^{\rho-1} F(\xi, a)}{R'(a)(\xi - a)^2} = \sum_{\alpha, \beta} \varkappa_{\alpha\beta}\, g_\alpha(\xi)\, h_\beta(t),$$

according to (6) and (7). Hence follows
$$\chi_\beta(\xi) = \sum_\alpha \varkappa_{\alpha\beta}\, g_\alpha(\xi). \tag{14}$$

Thus we obtain the

Theorem II.

The simultaneous periods $2\omega_\alpha$, $2\eta_\alpha$ *of the integrals* w_α, $w_{\rho+\alpha}$ *satisfy also the differential equations*

$$\frac{\partial\eta_\alpha}{\partial a} = -\sum_\beta \lambda_{\alpha\beta}\,\omega_\beta - \sum_\beta \varkappa_{\alpha\beta}\,\eta_\beta, \tag{B}$$

the quantities $\varkappa_{\alpha\beta}$ *and* $\lambda_{\alpha\beta}$ *being defined by* (7) *and* (12).

Let now

$$\left.\begin{matrix} 2\omega_{\alpha\beta}, & 2\omega'_{\alpha\beta} \\ 2\eta_{\alpha\beta}, & 2\eta'_{\alpha\beta} \end{matrix}\right\} \tag{15}$$

be a canonical system of periods of the integrals w_α and $w_{\rho+\alpha}$ respectively. The differential equations satisfied by these periods, according to (A) and (B), may be written in an abbreviated form by the use of the notations of the theory of matrices. Write in a general way

$$M = \left\{ m_{\alpha\beta} \right\},$$

the first index always referring to the row, the second to the column; let further $\overline{M}$ denote, as usual, the matrix derived from M by interchanging the rows with the columns ("transverse of M"). And write finally

$$\left\{\frac{\partial\omega_{\alpha\beta}}{\partial a}\right\} = \delta\Omega, \qquad \left\{\frac{\partial\omega'_{\alpha\beta}}{\partial a}\right\} = \delta\Omega',$$

$$\left\{\frac{\partial\eta_{\alpha\beta}}{\partial a}\right\} = \delta\mathrm{H}, \qquad \left\{\frac{\partial\eta'_{\alpha\beta}}{\partial a}\right\} = \delta\mathrm{H}',$$

$$\Gamma = \left\{\frac{2g_\alpha(a)\,g_\beta(a)}{R'(a)}\right\}.$$

With these notations and with Cayley's agreements concerning multiplication and addition of matrices, the $4\rho^2$ differential equations obtained by applying (A) and (B) successively to the 2ρ period-paths, may be combined into the following four matrix equations:

$$\left.\begin{aligned} \delta\Omega &= \overline{\mathrm{K}}\Omega - \Gamma\mathrm{H}, \\ \delta\Omega' &= \overline{\mathrm{K}}\Omega' - \Gamma\mathrm{H}', \end{aligned}\right\} \tag{A'}$$

$$\left.\begin{aligned} \delta\mathrm{H} &= -\Lambda\Omega - \mathrm{K}\mathrm{H}, \\ \delta\mathrm{H}' &= -\Lambda\Omega' - \mathrm{K}\mathrm{H}'. \end{aligned}\right\} \tag{B'}$$

§4.—*The Partial Differential Equation for the Θ-Functions: First Proof.*

Weierstrass' function*

$$\Theta(u_1, u_2, \ldots, u_\rho; m, n)$$

associated with the canonical system of integrals w_α, $w_{\rho+\alpha}$ and the canonical system of periods (15) is defined as follows:

Let the three matrices A, B, T be defined by the equations

$$2\mathrm{A}\Omega = \mathrm{H}, \quad 2\mathrm{B}\Omega = 1, \quad 2\mathrm{B}\Omega' = \mathrm{T}. \tag{16}$$

Since†

$$\overline{\Omega}\mathrm{H} = \overline{\mathrm{H}}\Omega, \quad \Omega\overline{\Omega}' = \Omega'\overline{\Omega},$$

it follows that

that is

$$\left.\begin{aligned} \overline{\mathrm{A}} &= \mathrm{A}, & \overline{\mathrm{T}} &= \mathrm{T}, \\ a_{\beta\alpha} &= a_{\alpha\beta}, & \tau_{\beta\alpha} &= \tau_{\alpha\beta}. \end{aligned}\right\} \tag{27}$$

Then

$$\Theta(u_1, u_2, \ldots, u_\rho; m, n)$$
$$= \sum_{\nu_1, \nu_2, \ldots, \nu_\rho} e^{\phi(u_1, u_2, \ldots, u_\rho;\, \nu_1 + n_1, \nu_2 + n_2, \ldots, \nu_\rho + n_\rho) + 2\pi i \sum_\alpha m_\alpha (\nu_\alpha + n_\alpha)}, \tag{18}$$

where

$$\phi(u_1 \ldots u_\rho, \nu_1 \ldots \nu_\rho) = \sum_{\alpha, \beta} a_{\alpha\beta} u_\alpha u_\beta + 2\pi i \sum_{\alpha, \beta} b_{\alpha\beta} \nu_\alpha u_\beta + \pi i \sum_{\alpha, \beta} \tau_{\alpha\beta} \nu_\alpha \nu_\beta,$$

and the indices $\nu_1, \nu_2, \ldots, \nu_\rho$ take independently all integer values from $-\infty$ to $+\infty$.

The object of the present § is to express the derivative of Θ with respect to a branch-point a in terms of Θ and its first and second derivatives with respect to the u_α's. Since a is implicitly contained in the quantities $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$, we have‡

$$\frac{\partial\Theta}{\partial a} = \sum_\alpha \frac{\partial\Theta}{\partial a_{\alpha\alpha}} \frac{\partial a_{\alpha\alpha}}{\partial a} + \sum_{(\alpha, \beta)} \frac{\partial\Theta}{\partial a_{\alpha\beta}} \frac{\partial a_{\alpha\beta}}{\partial a} + \sum_{(\alpha, \beta)} \frac{\partial\Theta}{\partial b_{\alpha\beta}} \frac{\partial b_{\alpha\beta}}{\partial a}$$
$$+ \sum_\alpha \frac{\partial\Theta}{\partial \tau_{\alpha\alpha}} \frac{\partial \tau_{\alpha\alpha}}{\partial a} + \sum_{(\alpha, \beta)} \frac{\partial\Theta}{\partial \tau_{\alpha\beta}} \frac{\partial \tau_{\alpha\beta}}{\partial a}. \tag{19}$$

* "Lectures on Hyperelliptic Functions," and Schottky, "Abel'sche Functionen von drei Variabeln," §1; Baker, "Abel's Theorem," etc., Art. 189.

† These are, in matrix form, two of the well-known bilinear relations between the periods (15); see Baker, Art. 140.

‡ Compare for the notation end of §1.

a). *The partial derivatives of* Θ *with respect to* $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$: By differentiating the series (18) on the one hand with respect to the quantities $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$, on the other hand, with respect to the u_α's, the following theorem is easily verified:

Theorem III.

Considered as a function of the quantities

$$u_\alpha, \; a_{\alpha\beta}, \; b_{\alpha\beta}, \; \tau_{\alpha\beta},$$

Weierstrass' function

$$\Theta(u_1, u_2, \ldots, u_\rho; \; m, n)$$

satisfies the following partial differential equations:

$$\left.\begin{aligned}
&\frac{\partial\Theta}{\partial a_{\alpha\alpha}} = u_\alpha^2 . \Theta, \quad \frac{\partial\Theta}{\partial a_{\alpha\beta}} = 2u_\alpha u_\beta . \Theta, \qquad \beta \neq \alpha\\
&u_\beta \frac{\partial\Theta}{\partial u_\gamma} = 2\Theta . \sum_\alpha a_{\alpha\gamma} u_\alpha u_\beta + \sum_\alpha b_{\alpha\gamma} \frac{\partial\Theta}{\partial b_{\alpha\beta}}\\
&\frac{\partial^2\Theta}{\partial u_\gamma \partial u_\delta} = 2\Theta . \Big[a_{\gamma\delta} + 2\sum_{\alpha,\beta} a_{\alpha\gamma} a_{\beta\delta} u_\alpha u_\beta\Big] + 2\sum_{\alpha,\beta} a_{\gamma\alpha} b_{\beta\delta} \frac{\partial\Theta}{\partial b_{\beta\alpha}}\\
&\qquad + 2\sum_{\alpha,\beta} a_{\delta\beta} b_{\alpha\gamma} \frac{\partial\Theta}{\partial b_{\alpha\beta}} + 2\pi i \Big[2\sum_\alpha b_{\alpha\gamma} b_{\alpha\delta} \frac{\partial\Theta}{\partial\tau_{\alpha\alpha}} + {\sum_{\alpha,\beta}}' b_{\alpha\gamma} b_{\beta\delta} \frac{\partial\Theta}{\partial\tau_{\alpha\beta}}\Big],
\end{aligned}\right\} \text{(C)}.$$

where ${\sum\limits_{\alpha,\beta}}'$ *indicates that in the summation the terms for which* $\beta = \alpha$ *are to be omitted.*

The three systems of equations contained in (C) can again be written in the form of three matrix equations. Let

$$m_{\alpha\beta} \begin{cases} \dfrac{\partial\Theta}{\partial a_{\alpha\beta}} & \beta \neq \alpha \\ 2\dfrac{\partial\Theta}{\partial a_{\alpha\alpha}} & \beta = \alpha \end{cases}; \qquad n_{\alpha\beta} = \begin{cases} \dfrac{\partial\Theta}{\partial \tau_{\alpha\beta}} & \beta \neq \alpha \\ 2\dfrac{\partial\Theta}{\partial \tau_{\alpha\alpha}} & \beta = \alpha \end{cases}, \qquad (20)$$

$$P = \left(\frac{\partial\Theta}{\partial b_{\alpha\beta}}\right), \quad U = \left(u_\alpha u_\beta\right), \quad V = \left(u_\alpha \frac{\partial\Theta}{\partial u_\beta}\right), \quad W = \left(\frac{\partial^2\Theta}{\partial u_\alpha \partial u_\beta}\right),$$

and observe that $\overline{M} = M$, $\overline{N} = N$, $\overline{U} = U$, $\overline{W} = W$. With these notations the three matrix equations replacing (C) are:

$$\left.\begin{aligned}
&M = 2\Theta . U,\\
&V = 2\Theta . UA + \overline{P}B,\\
&W = 2\Theta . (A - 2AUA) + 2(AV + \overline{V}A) + 2\pi i\, \overline{B}NB.
\end{aligned}\right\} \qquad (C')$$

In these equations Θ plays the part of a scalar factor; they can be immediately solved with respect to N and $\overline{P}$, and in order to obtain P we have only to remember that for any two matrices A and B the rules hold:

$$\overline{A+B}=\bar{A}+\bar{B}$$

and

$$\overline{AB}=\bar{B}\bar{A}.$$

b). *The derivatives of $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$ with respect to a* can be found by combining (A′) and (B′) with (16). The matrix equation

$$2A\Omega=H,$$

written non-symbolically, reads

$$2\sum_\beta a_{\alpha\beta}\,\omega_{\beta\gamma}=\eta_{\alpha\gamma},$$

hence

$$2\sum_\beta \frac{\partial a_{\alpha\beta}}{\partial a}\,\omega_{\beta\gamma}+2\sum_\beta a_{\alpha\beta}\,\frac{\partial\omega_{\beta\gamma}}{\partial a}=\frac{\partial\eta_{\alpha\gamma}}{\partial a},$$

and if we denote again

$$\delta A=\left|\frac{\partial a_{\alpha\beta}}{\partial a}\right|,\quad \delta B=\left|\frac{\partial b_{\alpha\beta}}{\partial a}\right|,\quad \delta T=\left|\frac{\partial \tau_{\alpha\beta}}{\partial a}\right|,$$

the last equation may be written in matrix form

$$2\delta A\,.\,\Omega+2A\,.\,\delta\Omega=\delta H,$$

similarly

$$\delta B\,.\,\Omega+\ B\,.\,\delta\Omega=0,$$
$$\delta\Omega\,.\,T+\ \Omega\,.\,\delta T=\delta\Omega'.$$

Substituting for $\delta\Omega$, $\delta\Omega'$, δH their values from (A′), (B′) and solving for δA, δB, δT, we obtain the

Theorem IV.

The expressions for the derivatives of the quantities $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\tau_{\alpha\beta}$ with respect to a branch-point a are exhibited in the following matrix equations:

$$\left.\begin{aligned}\delta A&=-\tfrac{1}{2}\Lambda-(KA+A\overline{K})+2A\Gamma A,\\ \delta B&=-B(\overline{K}-2\Gamma A),\\ \delta T&=\quad 2\pi i B\Gamma\overline{B}.\end{aligned}\right\}\qquad\text{(D)}$$

c). We now return to equation (19). Using the notations (20), we may write it

$$\frac{\partial\Theta}{\partial a}=\tfrac{1}{2}\sum_{\alpha,\,\beta}m_{\alpha\beta}\frac{\partial a_{\alpha\beta}}{\partial a}+\sum_{\alpha,\,\beta}\frac{\partial\Theta}{\partial b_{\alpha\beta}}\,\frac{\partial b_{\alpha\beta}}{\partial a}+\tfrac{1}{2}\sum_{\alpha,\,\beta}n_{\alpha\beta}\frac{\partial\tau_{\alpha\beta}}{\partial a}.$$

This expression can be made accessible to the methods of the theory of matrices by the following remark:

Let, for any square matrix A of ρ^2 elements, $\{A\}$ denote the *sum of the elements in the principal diagonal,** that is

$$\{A\} = \sum_{\alpha} a_{\alpha\alpha}.$$

From this definition of the symbol $\{A\}$ follow at once the following rules:

$$\left.\begin{aligned} \{\overline{A}\} &= \{A\}, \\ \{A+B\} &= \{A\} + \{B\}, \\ \{AB\} &= \sum_{\alpha,\beta} a_{\alpha\beta} b_{\beta\alpha} = \{BA\}. \\ &\text{From the last equation follows:} \\ \{ABC\} &= \{BCA\} = \{CAB\} \\ \text{and} \quad \sum_{\alpha,\beta} a_{\alpha\beta} b_{\alpha\beta} &= \{A\overline{B}\} = \{\overline{A}B\}. \end{aligned}\right\} \tag{21}$$

Our expression for $\dfrac{\partial\Theta}{\partial a}$ may therefore be written

$$\frac{\partial\Theta}{\partial a} = \{\tfrac{1}{2}\overline{M}.\delta A + \overline{P}.\delta B + \tfrac{1}{2}\overline{N}.\delta T\}.$$

If we substitute for M, N, P and δA, δB, δT their values from (C), (C′), (D), we obtain

$$\begin{aligned} \frac{\partial\Theta}{\partial a} = \{ &-\tfrac{1}{2}\Theta.U\Lambda - \Theta.A\Gamma + \Theta(UA\overline{K} - UKA) \\ &+ 2\Theta(AUA\Gamma - UA\Gamma A) + (2V\Gamma A - AV\Gamma - \overline{V}A\Gamma) + \tfrac{1}{2}W\Gamma - V\overline{K}\}. \end{aligned}$$

This expression simplifies considerably, if we apply the rules (21), and remember that $\overline{A} = A$, $\overline{U} = \overline{U}$, $\Gamma = \Gamma$:

$$\left.\begin{aligned} \{UA\overline{K}\} &= \{K\overline{A}\overline{U}\} = \{KAU\} = \{UKA\}, \\ \{AUA\Gamma\} &= \{UA\Gamma A\} = \{\Gamma AUA\}, \\ \{V\Gamma A\} &= \{AV\Gamma\} = \Gamma AV\} = \{\overline{V}\overline{A}\overline{\Gamma}\} = \{\overline{V}A\Gamma\}. \end{aligned}\right\} \tag{22}$$

---

* Since the above was written, I found that the operation $\{A\}$ has been studied by Taber, in his paper "On the Application to Matrices of any Order of the Quaternion Symbols S and V," Proc. London Math. Soc., vol. XXII, p. 67.

Thus $\frac{\partial\Theta}{\partial a}$ reduces to

$$\frac{\partial\Theta}{\partial a}=\{\tfrac{1}{2}\Theta\,.\,U\Lambda-\Theta\,.\,A\Gamma-V\overline{K}+\tfrac{1}{2}W\Gamma\},\tag{23}$$

and returning to non-symbolical notation we have the

Theorem V.

Weierstrass' most general Θ-function defined by (18) *satisfies the partial differential equation*

$$\frac{\partial\Theta}{\partial a}=-\tfrac{1}{2}\Theta\,.\Big[\sum_{\alpha,\,\beta}\lambda_{\alpha\beta}u_\alpha u_\beta+4\sum_{\alpha,\,\beta}\frac{g_\alpha(a)\,g_\beta(a)\,a_{\alpha\beta}}{R'(a)}\Big]$$
$$-\sum_{\alpha,\,\beta}\varkappa_{\alpha\beta}u_\alpha\frac{\partial\Theta}{\partial u_\beta}+\sum_{\alpha,\,\beta}\frac{g_\alpha(a)\,g_\beta(a)}{R'(a)}\,\frac{\partial^2\Theta}{\partial u_\alpha\,\partial u_\beta},\tag{E}$$

the coefficients $\varkappa_{\alpha\beta}$ and $\lambda_{\alpha\beta}$ being defined by (7) *and* (12).

§5.—*The Partial Differential Equations for the Θ-Functions: Second Proof.*

According to Weierstrass, the function $\Theta(u_1\,\ldots\,u_\rho;\ m,n)$ is connected with the function

$$\vartheta(v_1\,\ldots\,v_\rho;\ m,n)=\sum_{\nu_1,\nu_2\,\ldots\,\nu_\rho}e^{\pi i\sum\limits_{\alpha,\,\beta}\tau_{\alpha\beta}(\nu_\alpha+n_\alpha)(\nu_\beta+n_\beta)+2\pi i\sum\limits_{\alpha}(\nu_\alpha+n_\alpha)(v_\alpha+m_\alpha)}\tag{24}$$

by the relation

$$\Theta(u_1\,\ldots\,u_\rho;\ m,n)=e^{g(u_1,\,\ldots,\,u_\rho)}\,\vartheta(v_1,v_2,\,\ldots,\,v_\rho;\ m,n),\tag{25}$$

where

$$g(u_1\,\ldots\,u_\rho)=\sum_{\alpha,\,\beta}a_{\alpha\beta}u_\alpha u_\beta,\tag{26}$$

and the v_β's are determined by

$$u_\alpha=\sum_\beta 2\omega_{\alpha\beta}v_\beta.\tag{27}$$

The function ϑ satisfies the partial differential equations

$$\frac{\partial^2\vartheta}{\partial v_\alpha^2}=4\pi i\frac{\partial\vartheta}{\partial\tau_{\alpha\alpha}},\quad\frac{\partial^2\vartheta}{\partial v_\alpha\,\partial v_\beta}=2\pi i\frac{\partial\vartheta}{\partial\tau_{\alpha\beta}}.\qquad(\beta\neq\alpha)\tag{28}$$

Hence

$$\frac{\partial\Theta}{\partial a}=e^{g(u_1\,\ldots\,u_\rho)}\frac{\partial\vartheta}{\partial a}+\Theta\,.\,\frac{\partial g(u_1\,\ldots\,u_\rho)}{\partial a}.$$

a). Since a is implicitly contained in the v_α's as well as in the $\tau_{\alpha\beta}$'s, we

have

$$\frac{\partial \vartheta}{\partial a}=\sum_{\alpha}\frac{\partial \vartheta}{\partial a_{\alpha\alpha}}\frac{\partial \tau_{\alpha\alpha}}{\partial a}+\sum_{(\alpha,\beta)}\frac{\partial \vartheta}{\partial \tau_{\alpha\beta}}\frac{\partial \tau_{\alpha\beta}}{\partial a}+\sum_{\alpha}\frac{\partial \vartheta}{\partial v_\alpha}\frac{\partial v_\alpha}{\partial a},$$

or on account of (28),

$$\frac{\partial \vartheta}{\partial a}=\frac{1}{4\pi i}\sum_{\alpha,\beta}\frac{\partial^2 \vartheta}{\partial v_\alpha \partial v_\beta}\frac{\partial \tau_{\alpha\beta}}{\partial a}+\sum_{\alpha}\frac{\partial \vartheta}{\partial v_\alpha}\frac{\partial v_\alpha}{\partial a}.$$

The first term on the right-hand side may be written

$$\frac{1}{4\pi i}\left\{\delta T.\left|\frac{\partial^2 \vartheta}{\partial v_\alpha \partial v_\beta}\right|\right\}.$$

But from (27) follows

$$\left|\frac{\partial^2 \vartheta}{\partial v_\alpha \partial v_\beta}\right|=4\overline{\Omega}.\left|\frac{\partial^2 \vartheta}{\partial u_\alpha \partial u_\beta}\right|.\Omega.$$

Substituting this value and the value for δT from (D), and making use of (16) and (21), we obtain

$$\frac{1}{4\pi i}\sum_{\alpha,\beta}\frac{\partial^2 \vartheta}{\partial v_\alpha \partial v_\beta}\frac{\partial \tau_{\alpha\beta}}{\partial a}=\left\{\tfrac{1}{2}\left|\frac{\partial^2 \vartheta}{\partial u_\alpha \partial u_\beta}\right|.\Gamma\right\}. \tag{29}$$

b). On the other hand

$$\sum_{\alpha}\frac{\partial \vartheta}{\partial v_\alpha}\frac{\partial v_\alpha}{\partial a}=\sum_{\alpha,\beta}\frac{\partial \vartheta}{\partial u_\beta}2\omega_{\beta\alpha}\frac{\partial v_\alpha}{\partial a},$$

but

$$\frac{\partial u_\beta}{\partial a}=0=\sum_{\alpha}2\omega_{\beta\alpha}\frac{\partial v_\alpha}{\partial a}+\sum_{\alpha}2\frac{\partial \omega_{\beta\alpha}}{\partial a}v_\alpha,$$

$$\therefore\ \sum_{\alpha}\frac{\partial \vartheta}{\partial v_\alpha}\frac{\partial v_\alpha}{\partial a}=-2\sum_{\alpha,\beta}v_\alpha\frac{\partial \vartheta}{\partial u_\beta}.\frac{\partial \omega_{\beta\alpha}}{\partial a}=-2\left\{\delta\Omega.\left|v_\alpha\frac{\partial \vartheta}{\partial u_\beta}\right|\right\}.$$

From (27) follows

$$\left|v_\alpha\frac{\partial \vartheta}{\partial v_\beta}\right|=\tfrac{1}{2}\Omega^{-1}.\left|u_\alpha\frac{\partial \vartheta}{\partial u_\beta}\right|;$$

hence if we substitute for $\delta\Omega$ its value from (A′) and remember that $H\Omega^{-1}=2A$, we obtain

$$\sum_{\alpha}\frac{\partial \vartheta}{\partial v_\alpha}\frac{\partial v_\alpha}{\partial a}=\left\{(-\overline{K}+2\Gamma A).\left|u_\alpha\frac{\partial \vartheta}{\partial u_\beta}\right|\right\}. \tag{30}$$

c). Now from (25)

$$e^{g(u_1 \dots u_\rho)} \frac{\partial^2 \vartheta}{\partial u_\alpha \partial u_\beta} = \frac{\partial^2 \Theta}{\partial u_\alpha \partial u_\beta} - \frac{\partial \Theta}{\partial u_\alpha} \frac{\partial g}{\partial u_\beta} - \frac{\partial \Theta}{\partial u_\beta} \frac{\partial g}{\partial u_\alpha} - \Theta \cdot \frac{\partial^2 g}{\partial u_\alpha \partial u_\beta} + \Theta \cdot \frac{\partial g}{\partial u_\alpha} \frac{\partial g}{\partial u_\beta}.$$

But from the definition of $g(u_1 \dots u_\rho)$ follows

$$\left\{ \frac{\partial \Theta}{\partial u_\alpha} \frac{\partial g}{\partial u_\beta} \right\} = 2\overline{V}A, \quad \left\{ \frac{\partial g}{\partial u_\alpha} \frac{\partial \Theta}{\partial u_\beta} \right\} = 2AV,$$

$$\left\{ \frac{\partial^2 g}{\partial u_\alpha \partial u_\beta} \right\} = 2A, \quad \left\{ \frac{\partial g}{\partial u_\alpha} \frac{\partial g}{\partial u_\beta} \right\} = 4AUA.$$

Further,

$$e^{g(u_1 \dots u_\rho)} u_\alpha \frac{\partial \vartheta}{\partial u_\beta} = u_\alpha \frac{\partial \Theta}{\partial u_\beta} - \Theta \cdot u_\alpha \frac{\partial g}{\partial u_\beta},$$

and

$$\left\{ u_\alpha \frac{\partial g}{\partial u_\beta} \right\} = 2UA.$$

Substituting the values thus obtained for $\left\{ \frac{\partial^2 \vartheta}{\partial u_\alpha \partial u_\beta} \right\}$ and $\left\{ u_\alpha \frac{\partial \vartheta}{\partial u_\beta} \right\}$ in (29) and (30) and making use of the relations (22), we reach the result

$$e^{g(u_1 \dots u_\rho)} \frac{\partial \vartheta}{\partial a} = \{ -\overline{K}V + 2\Theta \cdot \overline{K}UA - 2\Theta \cdot AUA\Gamma - \Theta \cdot A\Gamma + \tfrac{1}{2} W\Gamma \}. \quad (31)$$

d). Finally,

$$\frac{\partial g}{\partial a} = \sum_{\alpha, \beta} \frac{\partial a_{\alpha\beta}}{\partial a} u_\alpha u_\beta = \{ \delta A \cdot U \},$$

therefore, replacing δA by its values from (D), we have

$$\Theta \cdot \frac{\partial g}{\partial a} = \Theta \cdot \{ -\tfrac{1}{2} \Lambda U - KAU - A\overline{K}U + 2A\Gamma AU \}. \quad (32)$$

Adding (31) and (32) and observing that

$$\{ \overline{K}UA \} = AUK \} = \{ KAU \} = \{ UKA \} = \{ A\overline{K}U \},$$

we reach the final result

$$\frac{\partial \Theta}{\partial a} = \{ -\tfrac{1}{2} \Theta \cdot \Lambda U - \Theta \cdot A\Gamma - \overline{K}V + \tfrac{1}{2} W\Gamma \},$$

in accordance with (23).

§6.—*The Partial Differential Equation for Wiltheiss' Function* $Th(u_1 \ldots . u_\rho)$.

If ω denote the determinant

$$|\omega_{\alpha\beta}| = \omega,$$

and $Adj\omega_{\alpha\beta}$ the minor of $\omega_{\alpha\beta}$ in this determinant, we have

$$\frac{\partial\omega}{\partial a} = \sum_{\alpha,\beta} \frac{\partial\omega_{\alpha\beta}}{\partial a} Adj\omega_{\alpha\beta} = \{\delta\Omega . \{Adj\omega_{\alpha\beta}\}\}.$$

But

$$\left(\frac{\overline{Adj\omega_{\alpha\beta}}}{\omega}\right) = \Omega^{-1},$$

therefore

$$\frac{\partial \log \omega}{\partial a} = \{\delta\Omega . \Omega^{-1}\} = \{\overline{K} - \Gamma H \Omega^{-1}\},$$

or

$$\frac{\partial \log \omega}{\partial a} = \{K\} - 2\{\Gamma A\}. \tag{33}$$

To obtain the value of $\{K\}$, observe first that $\{K\}$ remains invariant, if we pass from one set of integrals of the first kind, w_α, to another, $\bar{w}_\alpha$, the function $F(x, \xi)$ remaining the same.

For if*

$$g_\alpha(x) = \sum_\gamma c_{\alpha\gamma} \bar{g}_\alpha(x),$$

we have at the same time,

$$\bar{h}_\delta(\xi) = \sum_\beta c_{\beta\delta} h_\beta(\xi),$$

and if we write

$$\sum_{\alpha,\beta} \varkappa_{\alpha\beta} g_\alpha(x) h_\beta \xi = \sum_{\gamma,\delta} \bar{\varkappa}_{\gamma\delta} \bar{g}_\gamma(x) \bar{h}_\delta(\xi),$$

we obtain

$$\overline{K} = \bar{C} K \bar{C}^{-1}, \tag{34}$$

and therefore according to the rules (21),

$$\{\overline{K}\} = \{K\}. \tag{35}$$

---

* The $\bar{g}_\alpha^{(x)}(x)$'s are again supposed independent of a

Further, it follows from (7), if we make use of (6), that

$$\sum_\alpha \varkappa_{\alpha\beta}\, g_\alpha(x) = \tfrac{1}{2}\frac{g_\beta(x)}{x-a} - \frac{g_\beta(a)\, F(x, a)}{R'(a)(x-a)}$$

or since by (5),

$$\frac{F(x, a)}{(x-a)^2} = \tfrac{1}{2}\frac{R'(a)}{x-a} - 2\sum_\alpha g_{\rho+\alpha}(a)\, g_\alpha(x), \tag{36}$$

$$\sum_\alpha \varkappa_{\alpha\beta}\, g_\alpha(x) = \tfrac{1}{2}\frac{g_\beta(x) - g_\beta(a)}{x-a} + \frac{2g_\beta(a)}{R'(a)}\sum_\alpha g_{\rho+\alpha}(a)\, g_\alpha(x).$$

And if we choose, as we may according to the above remark, $g_\alpha(x) = x^{\alpha-1}$, we obtain

$$\varkappa_{\alpha\alpha} = \frac{2g_\alpha(a)\, g_{\rho+\alpha}(a)}{R'(a)},$$

therefore

$$\{K\} = \sum_\alpha \varkappa_{\alpha\alpha} = \frac{2}{R'(a)}\sum_\alpha g_\alpha(a)\, g_{\rho+\alpha}(a),$$

or if we put in (36) $x = a$:

$$\{K\} = -\frac{1}{2R'(a)}\left(\frac{\partial^2 F(x, a)}{\partial x^2}\right)_{x=a} \tag{37}$$

So far the function $F(x, \xi)$ was only subject to the conditions (2). We now introduce the special assumption that $F(x, \xi)$ *shall be the* $\rho + 1^{st}$ *polar of* $R(x)$ *with respect to* ξ (Klein); then

$$\{K\} = -\frac{\rho}{4(2\rho+1)}\frac{R''(a)}{R'(a)}, \tag{38}$$

$R''(x)$ denoting the second derivative of $R(x)$.

We thus reach the

Theorem VI.

Under the assumption that $F(x, \xi)$ is the $\rho + 1^{st}$ polar of $R(x)$ with respect to ξ, the logarithmic derivative of the determinant ω with respect to a is expressible as follows:

$$\frac{\partial \log \omega}{\partial a} = -\frac{\rho}{4(2\rho+1)}\frac{R''(a)}{R'(a)} - 4\sum_{\alpha,\beta}\frac{g_\alpha(a)\, g_\beta(a)\, a_{\alpha\beta}}{R'(a)}. \tag{F}$$

By combining this result with (E) we immediately obtain

Theorem VII.

Under the same assumption, Wiltheiss' function

$$Th(u_1 \ldots . u_\rho) = \left(\frac{\pi}{2}\right)^{\frac{\rho}{2}} \omega^{-\frac{1}{2}}\, \Theta(u_1 \ldots . u_\rho)$$

satisfies the differential equation

$$\frac{\partial\, Th}{\partial a} = -\frac{\rho}{8\,(2\rho+1)}\,\frac{R''(a)}{R'(a)} - \tfrac{1}{2}\, Th\,.\sum_{\alpha,\,\beta} \lambda_{\alpha\beta}\, u_\alpha\, u_\beta$$
$$-\sum_{\alpha,\,\beta} \varkappa_{\alpha\beta}\, u_\alpha \frac{\partial\, Th}{\partial u_\beta} + \frac{1}{R'(a)} \sum_{\alpha,\,\beta} \frac{\partial^2 Th}{\partial u_\alpha\, \partial u_\beta}\, g_\alpha(a)\, g_\beta(a)\,. \tag{G}$$

§7.—*The Partial Differential Equations for the Functions* $\sigma_{\phi_{\rho+1}\psi_{\rho+1}}$.

Suppose, as in the previous §, $F(x,\,\xi)$ to be the $\rho + 1^{\text{st}}$ polar of $R(x)$ with respect to ξ, and let

$$R(x) = \phi(x)\,.\,\psi(x)$$

be a decomposition of $R(x)$ into two factors of degree $\rho+1$, $\Theta_{\phi\psi}$ and $\sigma_{\phi\psi}$ the corresponding Θ- and σ-functions. Then $\Theta_{\phi\psi}(0,\,0,\,\ldots\ldots,\,0) \neq 0$ and

$$\sigma_{\phi\psi}(u_1\,\ldots\ldots\,u_\rho) = \frac{\Theta_{\phi\psi}(u_1\,\ldots\ldots\,u_\rho)}{\Theta_{\phi\psi}(0\,\ldots\ldots\,0)}\,. \tag{39}$$

If in (E) we put

$$u_1 = 0\,,\quad u_2 = 0\,\ldots\ldots\,u_\rho = 0\,,$$

we obtain

$$\frac{\partial \log \Theta(0_1\,\ldots\ldots\,0)}{\partial a} = -\,2\sum_{\alpha,\,\beta} \frac{g_\alpha(a)\,g_\beta(a)\,a_{\alpha\beta}}{R'(a)} + \sum_{\alpha,\,\beta} \frac{\partial^2 \log\Theta}{\partial u_\alpha\,\partial u_\beta}\Big)_0 \frac{g_\alpha(a)\,g_\beta(a)}{R'(a)}\,. \tag{40}$$

But for arbitrary parameters s, t*

$$\sum_{\alpha,\,\beta} \left(\frac{\partial^2 \log\Theta}{\partial u_\alpha\,\partial u_\beta}\right)_0 g_\alpha(s)\,g_\beta(t) = \frac{\phi(s)\,\psi(t) + \phi(t)\,\psi(s) - 2F(s,\,t)}{4\,(t-s)}\,. \tag{41}$$

Expand $\phi(s)\,\psi(t) + \phi(t)\,\psi(s)$ into Clebsch-Gordan's series, and put $t = s$:

$$\sum_{\alpha,\,\beta} \left(\frac{\partial^2 \log\Theta}{\partial u_\alpha\,\partial u_\beta}\right)_0 g_\alpha(s)\,g_\beta(s) = \frac{\rho\,(\rho+1)^2}{8\,(2\rho+1)}\,(\phi,\,\psi)_2, \tag{42}$$

$(\phi,\,\psi)_2$ denoting the second transvectant of $\phi(s)$ and $\psi(s)$.

Putting $s = a$ and combining (40) and (E), we obtain the

Theorem VIII.

The σ-function which belongs to the decomposition of $R(x)$ into the two factors of degree $\rho+1$:

$$R(x) = \phi(x)\,\psi(x)\,,$$

---

* See Bolza, American Journal of Mathematics, vol. XVI, p. 30.

satisfies the partial differential equation

$$\frac{\partial \sigma_{\phi\psi}}{\partial a} = -\tfrac{1}{2}\,\sigma_{\phi\psi}\,.\sum_{\alpha,\beta}\lambda_{\alpha\beta}\,u_\alpha\,u_\beta - \frac{\rho(\rho+1)^2}{8(2\rho+1)}\,\frac{f(a)}{R'(a)}\,\sigma_{\phi\psi}$$
$$-\sum_{\alpha,\beta}\varkappa_{\alpha\beta}\,u_\alpha\,\frac{\partial\sigma_{\phi\psi}}{\partial u_\beta} + \sum_{\alpha,\beta}\frac{\partial^2\sigma_{\phi\psi}}{\partial u_\alpha\,\partial u_\beta}\,\frac{g_\alpha(a)\,g_\beta(a)}{R'(a)}, \qquad \text{(H)}$$

where $f(x) = (\phi,\,\psi)_2 = (\phi\psi)^2\phi_x^{\rho-1}\psi_x^{\rho-1}$.

From (40) it is easy to derive Thomae's* expression for $d\log\Theta_{\phi\psi}(0\,\ldots\,.\,0)$ in terms of the branch-points. In fact, if a be a root of the factor $\phi(x)$, we obtain from (41) by first letting $s=a$ and then $t=a$:

$$\sum_{\alpha,\beta}\left(\frac{\partial^2\log\Theta}{\partial u_\alpha\,\partial u_\beta}\right)_0 g_\alpha(a)\,g_\beta(a) = \tfrac{1}{8}\,\psi(a)\,\phi''(a) - \frac{\rho}{8(2\rho+1)}\,R''(a),$$

and combining this result with (40) and (F), we obtain

$$\frac{\partial}{\partial a}\log\frac{\Theta(0\,\ldots\,.\,0)}{\omega^{\frac{1}{4}}} = \tfrac{1}{8}\,\frac{\phi''(a)}{\phi'(a)} = \tfrac{1}{8}\,\frac{\partial\log\Delta_\phi}{\partial a}, \qquad (43)$$

Δ_ϕ being the discriminant of ϕ.

Similarly, if b be a root of $\psi(x)$, we have

$$\frac{\partial}{\partial b}\log\frac{\Theta(0_1\,\ldots\,.\,0)}{\omega^{\frac{1}{4}}} = \tfrac{1}{8}\,\frac{\psi''(b)}{\psi'(b)} = \tfrac{1}{8}\,\frac{\partial\log\Delta_\psi}{\partial b}.$$

Hence
$$d\log\Theta(0\,\ldots\,.\,0) = d\log\omega^{\frac{1}{4}}\,\Delta_\phi^{\frac{1}{8}}\,\Delta_\psi^{\frac{1}{8}}, \qquad (44)$$

which is Thomae's result.

UNIVERSITY OF CHICAGO, *September 22d*, 1898.

---

* Crelle's Journal, Bd. 71. p. 201; compare also Schroeder, "Ueber den Zusammenhang der Hyperelliptischen σ- und ϑ-Functionen," §12–§15.